Beta levered and beta unlevered

Pablo Fernández*
PricewaterhouseCoopers Professor of Corporate Finance
IESE Business School
Camino del Cerro del Aguila 3. 28023 Madrid, Spain.
Telephone 34-91-357 08 09 Fax 34-91-357 29 13 e-mail: fernandezpa@iese.edu

ABSTRACT

We prove that in a world without leverage cost the relationship between the levered beta (β_L) and the unlevered beta (βu) is the No-cost-of-leverage formula: $\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$.

We also prove that the value of tax shields in a world with no leverage cost is the present value of the debt (D) times the tax rate (T) times the required return to the unlevered equity (Ku), discounted at the unlevered cost of equity (Ku): D T Ku / (Ku – g).

We then analyze 7 valuation theories proposed in the literature to estimate the relationship between the levered beta and the unlevered beta: Modigliani-Miller (1963), Myers (1974), Miles-Ezzell (1980), Harris-Pringle (1985), Damodaran (1994), No-cost-of-leverage and Practitioners. ? Without leverage costs the relationship between the betas is the No-cost-of-leverage formula. Only Damodaran provides consistent valuations once leverage costs are allowed for, but he introduces leverage costs in an ad hoc way.

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1. Introduction

This paper provides clear, theoretically sound, guidelines to evaluate the appropriateness of various relationships between the levered beta and the unlevered beta.

The relationship between the levered beta (β_L) and the unlevered beta (βu) is

[10] $\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$.

For constant growth companies, we prove that the value of tax shields in a world with no leverage cost is the present value of the debt (D) times the tax rate (T) times the required return to the unlevered equity (Ku), discounted at the unlevered cost of equity (Ku):

[32] VTS = D T Ku / (Ku – g)

Please note that it does not mean that the appropriate discount for tax shields is the unlevered cost of equity. We discount D T Ku, which is higher than the tax shield. As shown in Fernandez (2001) equation [32] is the difference of two present values.

The paper is organized as follows.

In Section 2, we derive the relationship between the levered beta and the unlevered beta for perpetuities in a world without leverage costs. This relationship is equation [10].

[10] $\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$

In Section 3, we revise the financial literature about the relationship between the levered beta and the unlevered beta. We prove that the relationship between the levered beta and the unlevered beta for growing perpetuities in a world without leverage costs is:

[36] $\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$

Note that [36] is equal to [10]. So, we may conclude that formula [10] is not only applicable for perpetuities, but also for growing perpetuities and for the general case in a world without leverage cost.

In Section 4 we analyze the 7 theories for perpetuities. We prove that five of the seven theories provide inconsistent results: Harris-Pringle (1985), Miles-Ezzell (1980) Modigliani-Miller (1963), Myers (1974), and Practitioners. No-cost-of-leverage is the method to use in a world without leverage costs. Damodaran provides us acceptable results in a world with leverage costs (although he introduces leverage costs in an ad hoc way)

Our conclusions are in Section 5.

Appendix 1 contains the dictionary of the initials used in the paper, and Appendix 2 the main valuation formulas according to the seven valuation theories that we analyze.

2. Relationship between the levered beta and the unlevered beta for perpetuities in a world without leverage costs

The value of tax shields for perpetuities in a world without leverage cost is DT.

[1] VTS = Value of tax shields = DT

Many authors report this result. For example, Brealey and Myers (2000), Modigliani and Miller (1963), Taggart (1991), Copeland et al. (2000), and Fernandez (2001). Fernandez (2001) proves [1] computing the difference between the present value of the taxes paid in the unlevered company and the present value of the taxes paid in the levered company (GL).

But one problem of equation [1] is that DT can be understood as = D α T / α. At first glance, α can be anything, related or unrelated to the company that we are valuing. In Section 3 it will be seen that Modigliani and Miller (1963) assume that α is risk-free rate (R_F). Myers (1974) assumes that α is the cost of debt (Kd). Fernandez (2001) shows that the value of tax shields is the difference between the present values of the taxes paid by the unlevered company and the taxes paid by the levered company and proves that the correct α is the required return to unlevered equity (Ku). Fernandez (2001) also proves that for growing companies both Modigliani and Miller (1963) and Myers (1974) provide inconsistent results.

The formula for the adjusted present value [2] indicates that the value of the debt today (D) plus that of the equity (E) of the levered company is equal to the value of the equity of the unlevered company (Vu) plus the value of tax shields due to interest payments (VTS).

[2] E + D = Vu + VTS

Knowing the value of tax shields (DT), and considering that Vu = FCF/Ku, we may rewrite equation [2] as [3]

[3] $E + D (1 – T) = FCF/Ku$

Taking into consideration that the relationship between ECF and FCF for perpetuities is:

[4] $FCF = ECF + D\ Kd (1 – T)$, and that E = ECF/Ke, we find that the relationship between the required return to assets (Ku) and the required return to equity (Ke) in a world without leverage costs is:

[5] $Ke = Ku + (Ku – Kd) D (1 – T) /E$

The formulas relating the betas with the required returns are:

[6] $Ke = R_F + \beta_L P_M$

[7] $Ku = R_F + \beta u\ P_M$

[8] $Kd = R_F + \beta d\ P_M$

R_F is the risk-free rate and P_M is the market risk premium. Substituting [6], [7] and [8] in [5], we get:

[9] $R_F + \beta_L P_M = R_F + \beta u\ P_M + (R_F + \beta u\ P_M – R_F – \beta d\ P_M) D (1 – T) /E$

Then, the relationship between the beta of the levered equity (β_L), the beta of the unlevered equity (βu) and the beta of debt (βd) for <u>perpetuities</u> in a world <u>without leverage costs</u> is:

[10] $\beta_L = \beta u + (\beta u – \beta d) D (1 – T) /E$

3. Literature review: 7 main theories

There is a considerable body of literature on the discounted cash flow valuation of firms. We will now discuss the most salient papers, concentrating particularly on those that proposed different expressions for the relationship between levered beta and unlevered beta.

Before discusing the theories, it is useful to get the relationship between Ke, Ku, Kd, E, D, VTS and g (growth) for growing perpetuities. As Vu = FCF / (Ku – g), we can rewrite equation [2] as

[11] $E + D = FCF / (Ku – g) + VTS$

The relationship between the equity cash flow and the free cash flow is

[12] $FCF = ECF + D\ Kd\ (1 – T) – g\ D$

By substituting [12] in [11], we get:

[13] $E + D = [ECF + D\ Kd\ (1 – T) – g\ D] / (Ku – g) + VTS$

As the relationship between the equity cash flow (ECF) and the equity value is ECF = E (Ke – g) we may rewrite [13] as:

[14] $E + D = [E\ (Ke – g) + D\ Kd\ (1 – T) – g\ D] / (Ku – g) + VTS$

Multiplying both sides of equation [14] by (Ku – g) we get:

[15] $(E + D)\ (Ku – g) = [E\ (Ke – g) + D\ Kd\ (1 – T) – g\ D] + VTS\ (Ku – g)$

Eliminating – g (E + D) on both sides of the equation [15]:

[16] $(E + D)\ Ku = [E\ Ke + D\ Kd\ (1 – T)] + VTS\ (Ku – g)$

Equation [16] may be rewritten as:

[17] $D\ [Ku – Kd\ (1 – T)] – E\ (Ke – Ku) = VTS\ (Ku – g)$

In the constant growth case, the relationship between Ke, Ku, Kd, E, D, VTS and g is Equation [17]:

Modigliani and Miller (1958) studied the effect of leverage on the firm's value. In the presence of taxes and for the case of a perpetuity, they calculate the value of tax shields (VTS) by discounting the present value of the tax savings due to interest payments of a risk-free debt ($T\ D\ R_F$) at the risk-free rate (R_F). Their first proposition, with taxes, is transformed into Modigliani and Miller (1963, page 436, formula 3):

[18] $E + D = Vu + PV[R_F;\ D\ T\ R_F] = Vu + D\ T$

DT is the value of tax shields (VTS) for a perpetuity. This result is the same as our equation [1]. But as will proven later on, this result is only correct for perpetuities. Discounting the tax savings due to interest payments of a risk-free debt at the risk-free rate provides inconsistent results for growing companies. Modigliani and Miller' purpose was to illustrate the tax impact of debt on value. They never addressed the issue of the riskiness of the taxes and only treated perpetuities. Later on, it will be seen that if we relax the no-growth assumption, then new formulas are needed.

For a perpetuity, the relationship between levered beta and unlevered beta implied by [18] is [10] as we have seen in section 2. But for a growing perpetuity, the value of tax shields for a growing perpetuity, according to Modigliani and Miller (1963), is:

[19] $VTS = D\ T\ R_F / (R_F – g)$

Substituting [19] in [17], we get:

$D\ [Ku – Kd\ (1 – T)] – E\ (Ke – Ku) = D\ T\ R_F\ (Ku – g) / (R_F – g)$

Then, the relationship between the levered and the unlevered required return to equity according to Modigliani and Miller (1963) is:

[20] $Ke = Ku + (D / E)\ [Ku – Kd\ (1 – T) – T\ R_F\ (Ku – g) / (R_F – g)] =$

$= Ku + (D / E)\ [Ku – Kd\ (1 – T) – VTS\ (Ku – g) / D]$

And the relationship between levered beta and unlevered beta is

[21] $\beta_L = \beta u + (D / E)\ [\beta u – \beta d + (T\ Kd / P_M) – VTS\ (Ku – g) / (D\ P_M)]$

Myers (1974) introduced the APV (adjusted present value). According to it, the value of the levered firm is equal to the value of the firm with no debt (Vu) plus the present value of the tax saving due to the payment of interest (VTS). Myers proposes calculating the VTS by discounting the tax savings (D T Kd) at the cost of debt (Kd). The argument is that the risk of the tax saving arising from the use of debt is the same as the risk of the debt. Then, according to Myers (1974):

[22] VTS = PV [Kd; D T Kd]

It is easy to deduct that the relationship between levered beta and unlevered beta implied by [22] for growing perpetuities is [23]:

[23] $\beta_L = \beta u + (D / E) (\beta u - \beta d) [1 - T Kd / (Kd - g)]$

Luehrman (1997) recommends to value companies by using the Adjusted Present Value and calculates the VTS as Myers. This theory provides inconsistent results for companies other than perpetuities as will be shown later.

According to **Miles and Ezzell (1980)**, a firm that wishes to keep a constant D/E ratio must be valued in a different manner from the firm that has a preset level of debt. For a firm with a fixed debt target [D/(D+E)] they claim that the correct rate for discounting the tax saving due to debt (Kd T D) is Kd for the tax saving during the first year, and Ku for the tax saving during the following years.

The expression of Ke is their formula 22:

[24] Ke = Ku + D (Ku – Kd) [1 + Kd (1 – T)] / [(1 + Kd) E]

And the relationship between levered beta and unlevered beta implied by [24] is:

[25] $\beta_L = \beta u + (D / E) (\beta u - \beta d) [1 - T Kd / (1 + Kd)]$

Lewellen and Emery (1986) also claim that the most logically consistent method is Miles and Ezzell.

Harris and Pringle (1985) propose that the present value of the tax saving due to the payment of interest (VTS) should be calculated by discounting the tax saving due to the debt (Kd T D) at the rate Ku. Their argument is that the interest tax shields have the same systematic risk as the firm's underlying cash flows and, therefore, should be discounted at the required return to assets (Ku).

Then, according to Harris and Pringle (1985), the value of tax shields is:

[26] VTS = PV [Ku; D Kd T]

Substituting [26] for growing perpetuities in [17], we get:

[27] D [Ku – Kd (1 – T)] – E (Ke – Ku) = D Kd T

Substituting [26] for growing perpetuities in [17], we get:

[27] D [Ku – Kd (1 – T)] – E (Ke – Ku) = D Kd T

Then, the relationship between the levered and the unlevered required return to equity according to Harris and Pringle (1985) is:

[28] Ke = Ku + (D / E) (Ku – Kd)

And the relationship between levered beta and unlevered beta implied by [28] is:

[29] $\beta_L = \beta u + (D / E) (\beta u - \beta d)$

Ruback (1995) reaches formulas that are identical to those of Harris-Pringle (1985). **Kaplan and Ruback (1995)** also calculate the VTS "discounting interest tax shields at the discount rate for an all-equity firm". **Tham and Vélez-Pareja (2001)**, following an arbitrage argument, also claim that the appropriate discount rate for tax shields is Ku, the required return to unlevered equity.

Taggart (1991) gives a good summary of valuation formulas with and without personal income tax. He proposes that Miles & Ezzell's (1980) formulas should be used when the company adjusts to its target debt ratio once a year and Harris & Pringle's (1985) formulas when the company continuously adjusts to its target debt ratio.

Damodaran (1994, page 31) argues that if all the business risk is borne by the equity, then the formula relating the levered beta (β_L) with the asset beta (βu) is:

[30] $\beta_L = \beta u + (D / E)\ \beta u\ (1 - T)$.

It is important to note that formula [30] is exactly formula [10] assuming that $\beta d = 0$. One interpretation of this assumption is (see page 31 of Damodaran, 1994) that "all of the firm's risk is borne by the stockholders (i.e., the beta of the debt is zero)". But we think that, in general, it is difficult to justify that the debt has no risk and that the return on the debt is uncorrelated with the return on assets of the firm. We rather interpret formula [30] as an attempt to introduce some leverage cost in the valuation: for a given risk of the assets (βu), by using formula [30] we obtain a higher β_L (and consequently a higher Ke and a lower equity value) than with formula [10]. Equation [30] appears in many finance books and is used by many consultants and investment bankers.

In some cases it could be not so outrageous to give debt a beta of 0. But if this is the case, then the required return to debt is the risk-free rate.

Another way of calculating the levered beta with respect to the asset beta is the following:

[31] $\beta_L = \beta u\ (1+ D / E)$.

We will call this method the **Practitioners' method**, because it is often used by consultants and investment banks (One of the many places where it appears is Ruback, 1995, page 5). It is obvious that according to this formula, given the same value for βu, a higher βL (and a higher Ke and a lower equity value) is obtained than according to [10] and [30].

One should notice that formula [31] is equal to formula [30] eliminating the (1 – T) term. We interpret formula [31] as an attempt to introduce still higher leverage cost in the valuation: for a given risk of the assets (βu), by using formula [31] we obtain a higher β_L (and consequently a higher Ke and a lower equity value) than with formula [30].

It is important to note that Damodaran (1994) and Practitioners impose a cost of leverage, but they do so in an ad hoc way.

Inselbag and Kaufold (1997) argue that if the firm targets the dollar values of debt outstanding, the VTS is given by the Myers (1974) formula. However, if the firm targets a constant debt/value ratio, the VTS is given by the Miles and Ezzell (1980) formula.

Copeland, Koller y Murrin (2000) treat the Adjusted Present Value in their Appendix A. They only mention perpetuities and only propose two ways of calculating the VTS: Harris y Pringle (1985) and Myers (1974). They also claim that "the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct." And they conclude "we leave it to the reader's judgment to decide which approach best fits his or her situation." It is quite interesting to note that Copeland et al. (2000, page 483) only suggest Inselbag and Kaufold (1997) as additional reading on Adjusted Present Value.

We will consider an additional theory to calculate the value of tax shields. We label this theory the **No-Costs-Of-Leverage** formula because, as proven in Fernandez (2002), it is the only formula that provides consistent results. According to this theory, the VTS is the present value of D T Ku (not the interest tax shield) discounted at the unlevered cost of equity (Ku):

[32] VTS = PV[Ku; D T Ku]

Dividing both sides of equation [17] by D (debt value), we get:

[33] [Ku – Kd (1 – T)] – (E / D) (Ke – Ku) = (VTS / D) (Ku – g)

If (E / D) is constant, the left-hand side of equation [33] does not depend on growth (g) because for any growth rate (E / D), Ku, Kd, and Ke are constant. We know that for g = 0, VTS = DT (equation [1]). Then, equation [33] applied to perpetuities (g = 0) is:

[34] [Ku – Kd (1 – T)] – (E / D) (Ke – Ku) = T Ku

Substracting [34] from [33] we get

0 = (VTS / D) (Ku – g) – T Ku, which is equation [32] applied to growing perpetuities:

Equation [32] is also the result of applying equation [10] to the general case.

Substituting equation [32] in [17], we get:

[35] Ke = Ku + (D / E) (1 – T) (Ku – Kd)

And the relationship between levered beta and unlevered beta implied by [35] is:

[36] $\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$

Note that [36] is equal to [10]. So, we may conclude that formula [10] is not only applicable for perpetuities, but also for growing perpetuities and for the general case in a world without leverage cost.

4. Analysis of the 7 theories for growing perpetuities

Table 1 reports relationship between levered beta and unlevered beta of the 7 theories for the case of growing perpetuities.

Table 1. Growing perpetuities. Relationship between levered beta and unlevered beta according to the 7 theories.

	Theories		Formula
	[1]		[2]
1	No-Costs-Of-Leverage	[36]	$\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$
2	Damodaran	[30]	$\beta_L = \beta u + (D / E)\ \beta u\ (1 - T)$.
3	Practitioners	[31]	$\beta_L = \beta u\ (1 + D / E)$.
4	Harris-Pringle	[29]	$\beta_L = \beta u + (D / E)\ (\beta u - \beta d)$
5	Myers	[23]	$\beta_L = \beta u + (D / E)\ (\beta u - \beta d)\ [1 - T\ Kd / (Kd - g)]$
6	Miles-Ezzell	[25]	$\beta_L = \beta u + (D / E)\ (\beta u - \beta d)\ [1 - T\ Kd / (1 + Kd)]$
7	Modigliani-Miller	[21]	$\beta_L = \beta u + (D / E)\ [\beta u - \beta d + (T\ Kd / P_M) - VTS\ (Ku - g) / (D\ P_M)]$

It may be seen that without growth, Myers and Modigliani-Miller formulae equal the No-Costs-Of-Leverage formula.

Both, Harris and Pringle (1985) and Miles and Ezzell (1980) formulae equal the No-Costs-Of-Leverage formula when T = 0 (no taxes). But when T > 0, both formulae provide a higher levered beta than the No-Costs-Of-Leverage formula. Then, we may conclude that both Harris and Pringle (1985) and Miles and Ezzell (1980) provide inconsistent results. They are not appropriate for valuing companies without leverage cost because the No-Costs-Of-Leverage formula is the right one. And they are not appropriate for valuing companies with leverage cost because when T = 0 (no taxes) leverage cost should also exist and both formulas provide us with the same value as the No-Costs-Of-Leverage formula. Fernandez (2001) shows that both Harris and Pringle (1985) and Miles and Ezzell (1980) provide inconsistent results looking at the value of tax shields.

In a world with leverage cost, equation [2] does not hold. A world with leverage cost is characterized by [37]:

[37] Vu + Gu = E + D + GL + LC

LC are the leverage cost. Leverage cost is the difference of the total value of the unlevered company (Vu + Gu) and that of the levered company (E + D + GL). The leverage cost have two main components:

- Increase of risk. It is obvious that the probability of default is higher for the levered company than for the unlevered company.
- Decrease of expected flows. It may be more difficult for the levered company than for the unlevered company to raise new funds and to benefit from growth opportunities.

Damodaran and Practitioners impose a cost of leverage, but they do so in an ad hoc way.

The finance literature tells us very little about how to calculate costs of leverage and how the magnitude of debt, the type of debt, taxes, market risk premium and other factors influence them. There is a need for further research on this area.

Obviously, the levered beta (β_L) should be higher than the unlevered beta (βu) because the equity cash flow is riskier than the free cash flow. From the relationships between β_L and βu of table 1, we may extract some major consequences that affect the validity of the theories. These consequences are summarized on table 2:

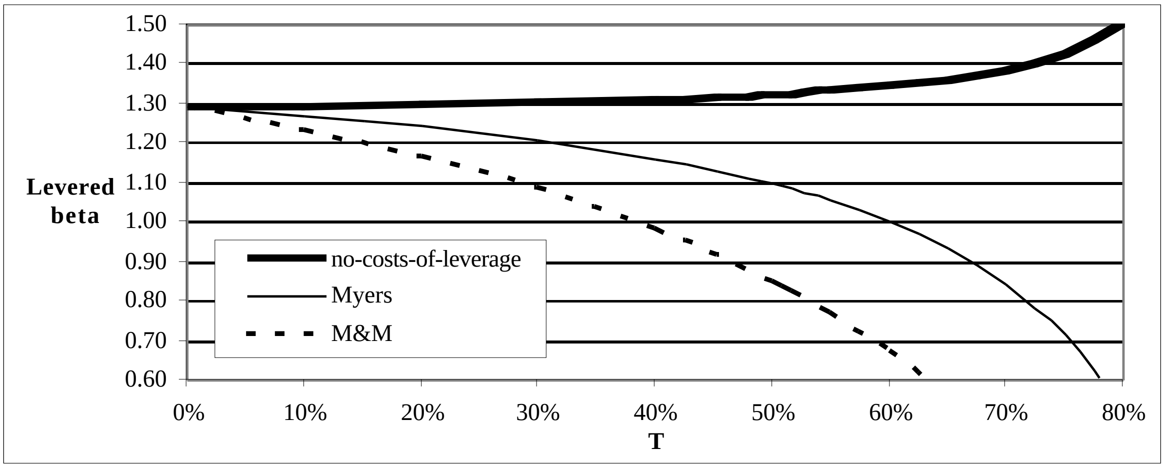
- The No-Costs-Of-Leverage formula always provides us with $\beta_L > \beta u$ because βu is always higher than βd.
- Myers provides us with the inconsistent result of β_L being lower than βu if the value of tax shields is higher than the value of debt. It happens when D T Kd / (Kd – g) > D, that is, when the growth rate is higher than the after-tax cost of debt: g > Kd (1 – T). Please note that in this situation, as the value of tax shields is higher than the value of debt, the equity (E) is worth more than the unlevered equity (Vu). This result makes no economic sense.
- Modigliani-Miller provides us with the inconsistent result of β_L being lower than βu if the value of tax shields is higher than D [Ku – Kd (1 – T)] / (Ku – g). It happens when the leverage, the tax rate, the cost of debt or the market risk premium are high.

Table 2. Problems of the candidate formulas in a world without leverage cost with constant growth: Levered beta may be lower than Unlevered beta

	Relationship between the levered beta and the unlevered beta	Levered beta < Unlevered beta
No-Costs-Of-Leverage	$\beta_L = \beta u + (\beta u - \beta d)\ D\ (1 - T) / E$	**never**
Myers	$\beta_L = \beta u + (D / E)\ (\beta u - \beta d)\ [1 - T\ Kd / (Kd - g)]$	If g > Kd (1 – T)
Modigliani-Miller	$\beta_L = \beta u + (D / E)\ [\beta u - \beta d + (T\ Kd / P_M) - - VTS\ (Ku - g) / (D\ P_M)]$	If g >R_F(Ku-Kd) (1-T) / [(Ku–Kd + T(Kd-R_F)]

The No-Costs-Of-Leverage formula does not present incongruencies. Figure 1 offers us a sensitivity analysis of the three theories for an example with constant growth. It may be seen that according to Myers and according to Modigliani-Miller $\beta_L < \beta u$, as had been predicted in table 4. Furthermore, according to Myers and Modigliani-Miller, β_L decreases when the tax rate increases. According to the No-Costs-Of-Leverage formula β_L increases when the tax rate increases.

Figure 1. Three theories to calculate the levered beta in a world without leverage costs
Unlevered beta = 1.0; Risk-free rate =12.0%; Market risk premium = 8%; Kd = 15.0%; g (growth) = 6.0%



We have already mentioned that an acceptable theory should provide us with a levered beta (β_L) higher than the unlevered beta (βu). It is also obvious that levered beta in a world with cost of leverage provided by the theory should be higher than the levered beta in a world without cost of leverage according to the No-Costs-Of-Leverage formula ($\beta_{L\mathbf{NCL}}$).

Table 3 shows that Damodaran provides us acceptable results (although he introduces leverage costs in an ad hoc way) and Practitioners provides us with inconsistent results.

Table 3. Candidate formulas to calculate the VTS in a world with cost of leverage and with constant growth.

Expected result	$\beta_L > \beta u$ **and** $\beta_L > \beta_{L\mathbf{NCL}}$	
Damodaran	OK	Acceptable
Practitioners	**NO** for high tax rates and for high unlevered betas	Inconsistent results

β_{LNCL} is the levered beta according to the No-Costs-Of-Leverage formula

Figures 2 and 3 offer us a sensitivity analysis of the two theories for an example with constant growth. It can be seen on figure 2 that the practitioners formula provides us with negative values of levered beta when the tax rate is higher than 65%. In figure 3 we can see that the the practitioners formula provides us with negative values of levered beta when the unlevered beta is around 2.5.

Figure 2. Two theories to calculate the levered beta in a world with leverage cost.
Unlevered beta = 1; Risk-free rate =12.0%; Market risk premium = 8.0%; g (growth) = 7.0%; Kd = 15%

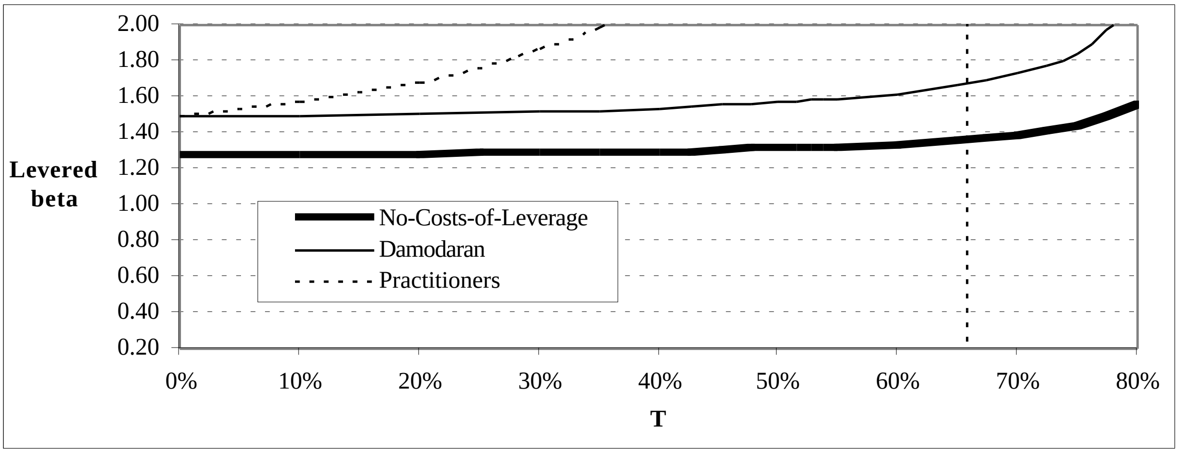
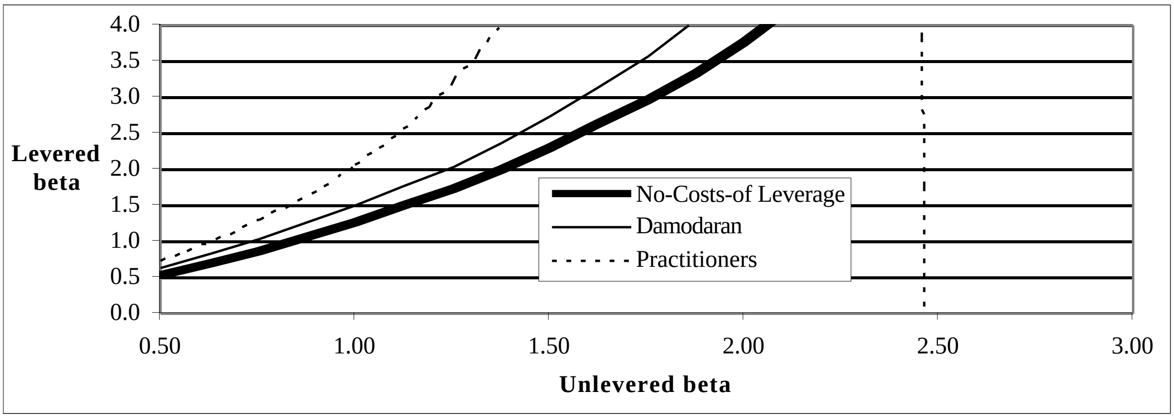


Figure 3. Two theories to calculate the levered beta in a world with leverage cost.
Risk-free rate =12.0%; T = 40.0%; Market risk premium = 8.0%; g (growth) = 9.0%; Kd = 15%



5. Conclusions

This paper provides clear, theoretically sound, guidelines to evaluate the appropriateness of various relationships between the levered beta and the unlevered beta.

For constant growth companies, we prove that the relationship between the levered beta (β_L) and the unlevered beta (βu) in a world with no leverage cost is

[36] $\beta_L = \beta u + (\beta u - \beta d)\, D\, (1 - T) / E.$

For constant growth companies, we prove that the value of tax shields in a world with no leverage cost is the present value of the debt (D) times the tax rate (T) times the required return to the unlevered equity (Ku), discounted at the unlevered cost of equity (Ku):

[32] $VTS = D\ T\ Ku / (Ku - g)$

We also prove that: Harris and Pringle (1985), Modigliani and Miller (1963), Myers (1974), Miles and Ezzell (1980), and practitioners provide inconsistent results.

Damodaran (1994) provides consistent results in a world with leverage cost, but he introduces leverage cost in an ad hoc manner.

In order to operationalize a valuation, very often one begins with assumptions of βd and β_L, not with βu. βu has to be inferred from βd and β_L. Which theories allow us to calculate βu? Without leverage costs the relationship between the betas is equation [36], that may be rewritten as

[36] $\beta u = [E\ \beta_L + \beta d\ D\ (1 – T)] / [E + D\ (1 – T)]$

We do not know for sure how to calculate the leverage costs, but the only theory that provides consistent results with leverage costs (although it introduces leverage costs in an ad hoc way) is equation [30], that may be rewritten as being

[30] $\beta u = E\ \beta_L / [E + D\ (1 – T)]$

APPENDIX 1
Dictionary

βd = Beta of debt

β_L = Beta of levered equity

βu = Beta of unlevered equity = beta of assets

D = Value of debt

E = Value of equity

ECF = Equity cash flow

FCF = Free cash flow

g = Growth rate of the constant growth case

I = interest paid = D Kd

Ku = Cost of unlevered equity (required return to unlevered equity)

Ke = Cost of levered equity (required return to levered equity)

Kd = Required return to debt = cost of debt

LC = Leverage cost

P_M = Market risk premium = E (R_M – R_F)

PV = Present value

R_F = Risk-free rate

T = Corporate tax rate

VTS = Value of the tax shields

Vu = Value of shares in the unlevered company

WACC = weighted average cost of capital

APPENDIX 2

Main valuation formulas
Market value of the debt = Nominal value

	No-cost-of-leverage	Damodaran (1994)	Practitioners
Ke	[35] $Ke = Ku + \frac{D(1-T)}{E}(Ku - Kd)$	$Ke = Ku + \frac{D(1-T)}{E}(Ku - R_F)$	$Ke = Ku + \frac{D}{E}(Ku - R_F)$
$ß_L$	[36] $\beta_L = \beta u + \frac{D(1-T)}{E}(\beta u - \beta d)$	[30] $\beta_L = \beta u + \frac{D(1-T)}{E}\beta u$	[31] $\beta_L = \beta u + \frac{D}{E}\beta u$
WACC	$Ku\left(1 - \frac{DT}{E+D}\right)$	$Ku\left(1 - \frac{DT}{E+D}\right) + D\frac{(Kd - R_F)(1-T)}{(E+D)}$	$Ku - D\frac{R_F - Kd(1-T)}{(E+D)}$
VTS	[32] PV[Ku; DTKu]	PV[Ku; DTKu - D (Kd- R_F) (1-T)]	PV[Ku; DTKd - D(Kd- R_F)]

	Harris-Pringle (1985) Ruback (1995)	Myers (1974)	Miles-Ezzell (1980)
Ke	[28] $Ke = Ku + \frac{D}{E}(Ku-Kd)$	$Ke = Ku + \frac{D-DVTS}{E}(Ku-Kd)$	[24] $Ke = Ku + \frac{D}{E}(Ku-Kd)\left[1 - \frac{T\,Kd}{1+Kd}\right]$
$ß_L$	[29] $\beta_L = \beta u + \frac{D}{E}(\beta u - \beta d)$	[23] $\beta_L = \beta u + \frac{D-DVTS}{E}(\beta u - \beta d)$	[25] $\beta_L = \beta u + \frac{D}{E}(\beta u - \beta d)\left[1 - \frac{T\,Kd}{1+Kd}\right]$
WACC	$Ku - D\frac{Kd\,T}{(E+D)}$	$Ku - \frac{DVTS(Ku-Kd) + D\,Kd\,T}{(E+D)}$	$Ku - D\frac{Kd\,T}{(E+D)}\frac{1+Ku}{1+Kd}$
VTS	[26] PV[Ku; D T Kd]	[22] PV[Kd; D T Kd]	PV[Ku; D T Kd] (1+Ku)/(1+Kd_0)

	Modigliani-Miller
Ke	[20] $Ke = Ku + \frac{D}{E}\left[Ku - Kd(1-T) - (Ku-g)\frac{VTS}{D}\right]$*
$ß_L$	[21] $\beta_L = \beta u + \frac{D}{E}\left[\beta u - \beta d + \frac{TKd}{P_M} - \frac{VTS(Ku-g)}{D\,P_M}\right]$*
WACC	$\frac{D\,Ku - (Ku-g)\,VTS}{(E+D)}$ *
VTS	[19] PV[R_F; D T R_F]

* Valid only for growing perpetuities

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